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Lifshitz Tails for Acoustic Waves in Random Quantum Waveguide

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In this study, we consider acoustic operators in a random quantum waveguide. Precisely we deal with an elliptic operator in the divergence form on a random strip. We prove that the integrated density of states of the relevant operator exhibits Lifshitz behavior at the bottom of the spectrum. This result could be used to prove localization of acoustic waves at the bottom of the spectrum.

KEY WORDS: spectral theory, random operators, integrated density of states, Lifshitz tails, localization, waveguide

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1. INTRODUCTION

The study of quantum waves on quantum waveguide has gained much interest and has been intensively studied during the last years for their important physical consequences. The main reason is that they represent an interesting physical effect with important applications in nanophysical devices, but also in flat electromagnetic waveguide.[(33)]

Exner *et al.* have done many works in this field. They have obtained results in different contexts we quote.[(9,10,11,12)] Also in Refs. 19, 26 we have research conducted in this area; the first is given for the discrete case.

We notice that originally studied in the context of quantum mechanical electrons. In the present work we are inspired from the model given in Kleespies and Stollmann work,[(26)] for the Laplacian operator to study Lishitz tails in the context of classical waves in random quantum wave-guides. In spite of the clear similarities between localization of quantum mechanical electron localization and

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localization of classical waves, there are some significant differences, classical waves are harder to localize.[17] Indeed, a local change in a homogeneous medium cannot create localized eigenfunction for classical waves operators but it can certainly create localized states for Schrödinger operators i.e quantum mechanical models to study electron waves in disordered media. For random Schrödinger operators it is proved[26] that in the two dimensional case band edge localization occurs on the bottom of the spectrum. It is natural to ask whether the same kind of phenomenon can appear for classical waves such as acoustic waves (The answer is the main object of this work, and it is proved to be positive).

We consider the divergence operator of the following form,

$$H = -\nabla \rho^{-1} \cdot \nabla \tag{1.1}$$

Here ρ is 2×2 diagonal matrix

$$\rho = \begin{pmatrix} \varrho & 0 \\ 0 & \rho_0 \end{pmatrix} \tag{1.2}$$

We assume that ρ_0 is a positive constant and ϱ is a bounded measurable function which represents the density of the medium where the wave propagates on the x_1 direction. We assume

$$\mu_0 \leq \varrho \leq \rho_0;$$

for some positive constante μ_0.

The great interest of this operator, both from the physical and the mathematical point of view, is quite obvious and known.[42] Below we give a brief description of the origin of this operator.

1.1. The Acoustic Operator

An acoustic wave is governed by the following system:

$$(S1) \begin{cases} \kappa \dfrac{\partial p}{\partial t} = -\nabla \cdot u \\ \varrho \dfrac{\partial u}{\partial t} = -\nabla p. \end{cases}$$

Here at time t and position x, $p = p(x,t)$ represents the pressure, while $u(x,t)$ represents the velocity, $\kappa = \kappa(x)$ is the compressibility and $\varrho(x)$ is the mass density of the media at point x. From $(S1)$ one deduces that p satisfies the equation

$$\kappa \frac{\partial^2 p}{\partial t^2} = \nabla \cdot \frac{1}{\varrho} \nabla p. \tag{1.3}$$

We define the momentum potential $\psi = \psi(x, t)$ by $\varrho u = -\nabla\psi$. So, it follows from $(S1)$ that ψ satisfies,

$$\kappa\frac{\partial p}{\partial t} = \nabla \cdot \frac{1}{\varrho}\nabla\psi; \quad \text{and} \quad \frac{\partial\psi}{\partial t} = p. \tag{1.4}$$

Therefore ψ obeys the following second order partial differential equation

$$\kappa\frac{\partial^2\psi}{\partial t^2} = \nabla \cdot \frac{1}{\varrho}\nabla\psi. \tag{1.5}$$

Motivated by Eqs. (1.3), (1.4) and (1.5), we set

$$H = -\nabla \cdot \frac{1}{\varrho} \cdot \nabla = \sum_{i=1}^{d} \partial_{x_i} \frac{1}{\varrho(x)} \partial_{x_i}. \tag{1.6}$$

H is called the acoustic operator. It is a partial differential operator which is elliptic under more assumptions on ϱ. When we deal with random media we note the density by ϱ_ω and the operator by H_ω.

1.2. The Integrated Density of States

As this paper is devoted to the study of the behavior of the integrated density of states, we recall that it is defined as follows: We note by H_Λ the restriction of H to a cube $\Lambda \subset \mathbb{R}^d$, with self-adjoint boundary conditions. As H is elliptic, the resolvent of H_Λ is compact and consequently, the spectrum of H_Λ is discrete and made of isolated eigenvalues of finite multiplicity.[41] We define

$$N_\Lambda(E) = \frac{1}{\mid \Lambda \mid} \cdot \#\{\text{eigenvalues of } H_\Lambda \le E\}. \tag{1.7}$$

Here $|\Lambda|$ is the volume of Λ in the Lebesgue sense and $\#E$ is the cardinal of E.

It is shown that the limit of $N_\Lambda(E)$ when Λ tends to $\mathbb{R}^d$ exists and is independent of the boundary conditions. It is called the **integrated density of states** of H_ω (IDS as an acronym) and noted by $N(E)$. See Ref. 22, 40.

The question we are interested in here deals with the behavior of N at the bottom of the spectrum of H. Let us give a brief history of this subject. In 1964, Lifshitz[34] argued that, for a Schrödinger operator of the form $H = -\Delta + V_\omega$, there exists $c_1, c_2, \alpha > 0$ such that $N(E)$ satisfies the asymptotic:

$$N(E) \simeq c_1 \exp(-c_2(E - E_0)^{-\alpha}), \quad E \to E_0. \tag{1.8}$$

Here E_0 is the bottom of the spectrum of H. The behavior (1.8) is known as **Lifshitz tails** (for more details see part IV.9.A of Ref. 40), and α is the Lifshitz exponent. Usually such an exponent is of the form $-\frac{d}{2}$, where d is the dimension. We notice that the Lifshitz behavior is among the properties characterizing random operators.

Lifshitz also expected (1.8) at fluctuating edges inside the spectrum. We refer to this asymptotic by "*internal Lifshitz tails*".

The principal results known about Lifshitz tails are mainly shown for Schrödinger operators on the whole space (for continuous and discrete cases). (See Refs. 2, 22, 28, 40 and references therein).

Lifshitz tails for an operator of type (1.1), were the subject of previous works,[35,36] where we obtain the behavior of N at the internal band edges of the spectrum of (1.1). For the bottom of the spectrum it is known that when the operator (1.1), acts on the whole space, the IDS has a weyl asymptotic and decreases only polynomially.[37]

In Ref. 20, the authors derive regularity properties for the density of states in the Anderson model on a one-dimensional strip for potentials with singular continuous distributions and show that the density of states is infinitely differentiable.

An investigation of a family of Dirichlet Laplacians on randomly dented strips in $\mathbb{R}^2$; is considered in Ref. 26. They prove dense point spectrum with exponentially localized eigenfunctions near its fluctuation boundary at the bottom of the spectrum. The proof is related to the Lifshitz tails on this region of the spectrum.

1.3. Results and Discussion

1.3.1. The Model

Let D_0 be the strip $\mathbb{R} \times (0, D_{\max})$. Let $(\omega_\gamma)_{\gamma\in\mathbb{Z}}$ be a family of independent and identically distributed random variables taking values in $[0, d]$ for $0 < d < D_{\max}$. We denote by $(\mathbb{P}, \mathcal{F}, \Omega)$ the corresponding probability space and assume that
(A.1)

$$\lim_{\varepsilon\to 0} \frac{\log\log(\mathbb{P}\{\omega_0 \in (0, \varepsilon)\})}{\log \varepsilon} = 0, \tag{1.9}$$

and the mean value $m = E(\omega_0) = \int x d\mathbb{P} > 0$.

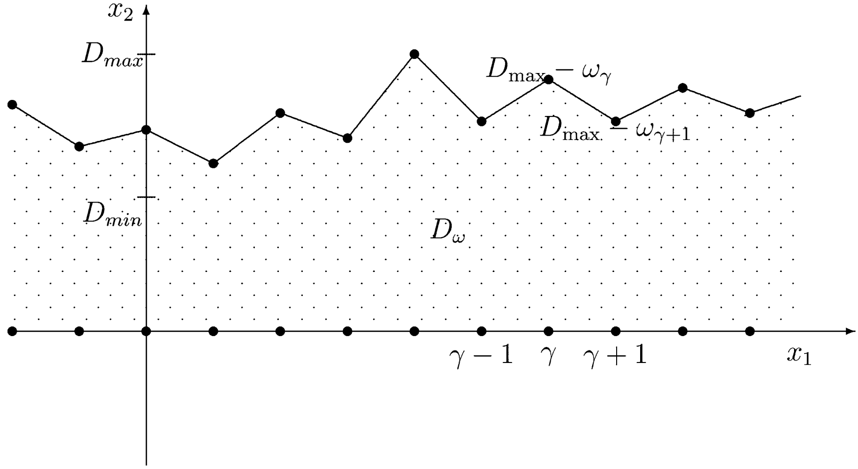
The random strip is defined as follows: The deviation of the width of the random strip from $D_{\max}$ is given by the γ-th coordinate ω_γ of $\omega \in \Omega$. For the family of points in $\mathbb{R}^2$; $\{(\gamma, (D_{\max} - \omega_\gamma))_{\gamma\in\mathbb{Z}}\}$ we consider $p(\omega)\colon \mathbb{R} \to [D_{\min}, D_{\max}]$ as a polygon joining these points. Let

$$D_\omega = \{(x_1, x_2) \in \mathbb{R}^2;\ 0 < x_2 < p(\omega)(x_1)\}.$$

This domain is drawn in the Picture 1:

Let $\mathcal{H}(\rho)$ be the following quadratic form defined as follow: for $u \in H_0^1(D_\omega) = \mathcal{D}(\mathcal{H}(\rho))$

$$\mathcal{H}(\rho)[u, u] = \int_{D_\omega} \rho^{-1} \nabla u(x) \overline{\nabla u(x)}\, dx.$$



Picture 1

Notice that here we have a family of quadratic forms acting on different domains. There is a family of random maps (φ_ω) that transform these different domains D_ω to the non-random domain, D_0 by dilatation (a change of variables). This transforms the randomness from the domain say to ρ which we denote by ρ_ω. Thus a random medium will be modeled by an ergodic random self-adjoint operator. Indeed the family of maps yield an equivalent quadratic form with domain $H_0^1(D_0)$

$$\mathcal{H}(\rho_\omega)[u, u] = \int_{D_0} \rho_\omega^{-1} \nabla u(x) \overline{\nabla u(x)}\, dx.$$

$\mathcal{H}(\rho_\omega)$ is a symmetrical, closed and positive quadratic form. Let H_ω be the restriction of the operator given by (1.1) to the domain D_ω with Dirichlet boundary conditions. H_ω is defined to be the self-adjoint operator associated to $\mathcal{H}(\rho_\omega)$.[41] Notice that $\rho_\omega = \rho(\varphi_\omega^{-1})$, consequently if we consider τ_γ the shift function i.e $(\tau_\gamma u)(x_1, x_2) = u(x_1 - \gamma, x_2)$. We assume that ϱ is $\mathbb{Z}$-periodic in x_1. This ensures that H_ω is a measurable family of self-adjoint operators and ergodic.[22,40] Indeed, $(\tau_\gamma)_{\gamma \in \mathbb{Z}}$ is a group of unitary operators on $L^2(D_0)$ and for $\gamma \in \mathbb{Z}$ we have

$$\tau_\gamma H_\omega \tau_{-\gamma} = H_{\tau_\gamma \omega}.$$

According to Refs. 22, 40 we know that there exists Σ, Σ_{pp}, Σ_{ac} and Σ_{sc} closed and non-random sets of $\mathbb{R}$ such that Σ is the spectrum of H_ω with probability one and such that if σ_{pp} (respectively σ_{ac} and σ_{sc}) design the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of H_ω, then $\Sigma_{pp} = \sigma_{pp}$, $\Sigma_{ac} = \sigma_{ac}$ and $\Sigma_{sc} = \sigma_{sc}$ with probability one.

2. RESULTS AND DISCUSSIONS

2.1. The Result

We notice that as $\mathbb{P}\{\omega_0 \in (0, \varepsilon)\} \neq 0$, one gets that D_ω contains rectangular boxes of length k in the x_1 direction and width $D_{\max} - \varepsilon$ for any $d > \varepsilon > 0$ and k large $\mathbb{P}$-almost surely. Using the fact that $\mu_0 \leq \varrho \leq \rho_0$ and the min-max principle by a comparison to the Laplacian and using the form of ρ one gets that for $\mathbb{P}$ almost every $\omega \in \Omega$.

$$\inf(\Sigma) = E_0 = \frac{\pi^2}{\rho_0 D_{\max}^2}. \tag{2.10}$$

Our study is in a neighborhood of this point.

Theorem 2.1. *Under the assumption* ***(A.1)****, the integrated density of states of* H_ω *satisfies:*

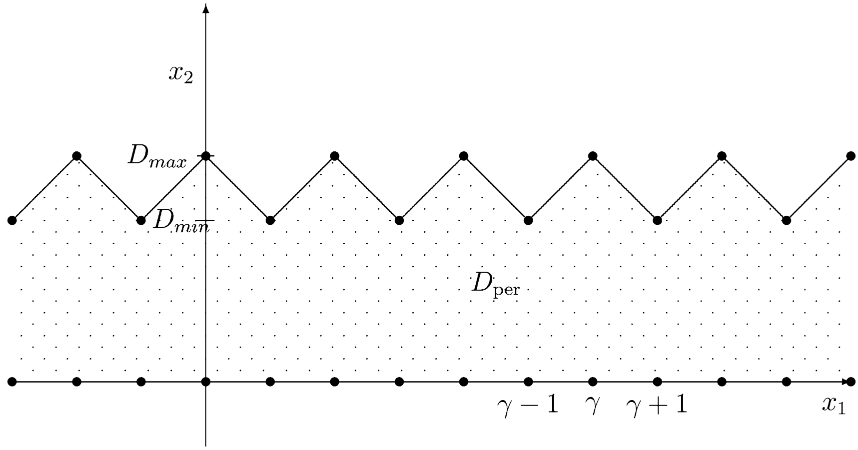
$$\lim_{\varepsilon \to 0} \frac{\log(|\log(N(E_0 + \varepsilon))|)}{\log \varepsilon} = -\frac{1}{2}.$$

Remark 2.2.

- By considering perturbation of a periodic medium (See picture 2) with ϱ is $\mathbb{Z}$-periodic in the x_1-direction one can get a spectrum with open gaps. Under adequate assumptions, the result is still true for internal band edges. This could be done using the periodic approximations and the reduction procedure.[28,35,36]
- We notice that here we have the Lifshitz exponent independent of ϱ compared to the long range and short range cases.[35,36] This is due to the fact that in the present case lifshitz exponent is due to the geometry of the the wave guide.
- In the present case we prove Lifshitz tails at the bottom of the spectrum. We mention that in the case when the the operator is considered on $L^2(\mathbb{R}^d)$, the bottom of the spectrum is 0, and the IDS decreases only polynomially fast at 0, see Refs. 31, 39.

Outline of the Proof: To prove Theorem 2.1, we prove a lower and an upper bounds on $N(E_0 + \varepsilon)$. The upper and lower bounds are proven separately and based on the following result (Theorem 5.25 p. 110 of Ref. 40).

$$\frac{1}{2k+1}\mathbb{E}\left\{N\left(H^D_{\Lambda_k}(\omega), E_0 + \varepsilon\right)\right\} \leq N(E_0 + \varepsilon) \leq \frac{1}{2k+1}\mathbb{E}\left\{N\left(H^N_{\Lambda_k}(\omega), E_0 + \varepsilon\right)\right\}. \tag{2.11}$$



Picture 2

Here $H^D_{\Lambda_k}(\omega)$ is the operator defined by (1.1) restricted to $\Lambda_k \times (0, D_{\max}) \cap D(\omega)$ with Dirichlet boundary condition also on the verticals parts, while $H^N_{\Lambda_k}(\omega)$ when we consider Neumann boundary condition on the vertical parts. We notice that (2.11) yields that we have to estimate an upper bound of

$$\frac{1}{2k+1} N\big(H^N_{\Lambda_k}(\omega), E\big) \cdot \mathbb{P}(E_0\big(H^N_{\Lambda_k}(\omega) \leq E_0 + \varepsilon\big).$$

The first factor can easily estimated by the weyl estimation $(C(E_0 + \varepsilon)^{\frac{1}{2}})$, while for the second we follow the standard perturbation domain arguments laid down in Ref. 28, 26.

To estimate the probability from above it is sufficient to use the fact that eigenvalues near E_0 are due to the littleness of random variables, which yields to the estimation of this rare event. We notice that by this, one follows the technique used in Ref. 26.

2.2. Application

Theorem 2.1 can be considered as a first step toward physically-motivated applications. One of them is the study of the so-called localization. This could be done under some additional assumptions on the behavior of the random variables in the vicinity of 0 or d.

We note that localization was initially given a spectral interpretation: pure point spectrum with exponentially decaying eigenfunctions (exponential localization). Intuitive physical notion of localization has also dynamical interpretation: the moments of a wave packet, initially localized both in space and in energy, should remain uniformly bounded under time evolution.

All the proofs of localization, except in the discrete case[1,19] for the multidimensional case, use the method of the multiscale analysis. This method was used for the first time by Fröhlich and Spencer[7] and Fröhlich, Martinalli, Spencer and Scoppolla,[14] at the beginning of the eighties and it knew many extensions and simplifications to lead to the form described in Ref. 8. This analysis makes it possible to obtain information on the operator in the whole space, starting from information on the operator restricted to cubes of finite size, (see **(P1)** and **(P2)** below).[8] Although it originally only gave exponential localization,[2,8,23,24] it was later shown to also yield dynamical localization by Germinet and De Bièvre,[15] strong dynamical localization for moments up to some finite order is given in Ref. 3. The bootstrap multiscale analysis of Germinet and Klein in Ref. 16 yield strong dynamical localization up to all orders in the Hilbert-Schmidt norm.

For the adoption of this technique to random strip see Ref. 26, 42. For the first initial length scale estimate it is given below:

Theorem 2.3. *Let $\theta \in \mathbb{R}^2$ and $E_0 > 0$ be the bottom of the spectrum of H_ω. Assume **(A.1)** hold. Then for any $\alpha > 1$, integer $p > 0$, for $k \in \mathbb{N}$ sufficiently large, one has*

$$\textbf{(P1)}\ \mathbb{P}\left(\left\{dist\left(\sigma(H^\theta_{\omega,\Lambda_{k^\alpha}}), E_0\right) \leq \frac{1}{k}\right\}\right) \leq \frac{1}{k^p}.$$

Where $H^\theta_{\omega,\Lambda_k}$ is the operator H_ω restricted to this box with θ-quasiperiodic boundary condition i.e with boundary condition $\varphi(x_1+\gamma, x_2) = e^{i\gamma\cdot\theta}\varphi(x_1, x_2)$ for any $\gamma \in 2k\mathbb{Z}$.

Theorem 2.3 is a consequence of Theorem 2.1. Indeed, using the Combes-Thomas estimate and the decomposition of resolvent we get **(P1)**. We omit details and refer the reader to Refs. 38, 43.

If we assume that H_ω satisfies a Wegner estimate[13,42] i.e for some $\alpha > 0$ and $n > 0$ for $E \in \mathbb{R}$ for $k \geq 1$ and $0 < \varepsilon < 1$, there exists $C(E) > 0$ such that one has

$$\textbf{(P2)}\ \mathbb{P}\left(\left\{dist\left(\sigma\left(H^\theta_{\omega,\Lambda_k}\right), E\right) \leq \varepsilon\right\}\right) \leq C(E)\cdot|\Lambda_k|^\alpha\cdot\varepsilon^n; \tag{2.12}$$

then, for E_0 using Theorem 2.3 for $\theta = 0$, we obtain the initial estimate to start a multi-scale analysis. This proves that the spectrum of H_ω is exponentially localized in some interval around the energy E_0 i.e that in some neighborhood of E_0 eigenfunctions associated to energies in that interval are exponentially-localized. More precisely we have

Theorem 2.4. *Let H_ω be as in (1.1) and restricted to D_ω. We assume that **(A.1)** and **(P2)** hold. There exists $\varepsilon_0 > 0$ such that*

(i) $\Sigma \cap [E_0, E_0 + \varepsilon_0] = \Sigma_{pp} \cap [E_0, E_0 + \varepsilon_0]$.

(ii) *an eigenfunction corresponding to an eigenvalue in $[E_0, E_0 + \varepsilon_0]$ decays exponentially.*

(iii) *for all $p > 0$,*

$$\mathbb{E}\{\sup_{t>0} \| |X|^p e^{itH_\omega} P_{[E_0, E_0+\varepsilon_0]}(H_\omega)\chi_K \| \} < +\infty.$$

Here $P_I(H_\omega)$ is the spectral projection on the interval I, χ_K is the characteristic function of K, K is a compact of $\mathbb{R}^d$ and X is the position operator.

To comment upon Theorem 2.4, let us consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = H_\omega u. \tag{2.13}$$

The solution of (2.13) is given[42] by

$$u(t, \cdot) = \cos(t\sqrt{H_\omega})u_0 + \sin(t\sqrt{H_\omega})u_1,$$

where $u_0 = u(0, \cdot)$ and $\sqrt{H_\omega}u_1 = (\partial_t u)(0, \cdot)$ denote the initial data.

A localized acoustic wave should be a finite energy solution of (2.13) with the property that almost all the wave's energy remains in a fixed bounded region of space at all times. Thus, if u_0 and u_1 are linear combinations of exponentially decaying eigenfunctions, $u(t)$ will be concentrated in some fixed ball for all times and the respective waves are localized.

By this, the result of Theorem 2.3 and therefore that of Theorem 2.4 is related to the behavior of the integrated density of states in the neighborhood of the so-called fluctuation boundary E_0.[27,40,43]

3. PRELIMINARY

Let us start this section by transforming the perturbation of the medium to a perturbation on the operator.

For $k \in 2\mathbb{N} + 1$ and $\gamma \in \mathbb{Z}$, we set $\Lambda_k(\gamma) = (\gamma - \frac{k}{2}, \gamma + \frac{k}{2})$ and $\Lambda_k = \Lambda_k(0)$. Let $f: \Lambda_k \to [D_{\min}, D_{\max}]$. For $0 \le t < \inf_{\Lambda_k} f$ be a measurable function. We consider the bounded domain of $\mathbb{R}^2$;

$$D_{t,k} = \{(x_1, x_2);\ x_1 \in \Lambda_k,\ 0 < x_2 < f(x_1) - t\}.$$

Remark 3.1. By the notation given at the beginning of this section we have

$$D_\omega = D_{0,\infty}$$

for

$$f_\omega : \mathbb{R} \to [D_{\min}, D_{\max}]$$
$$x_1 \mapsto \sum_{\gamma \in \mathbb{Z}} u_\gamma$$

and u_γ is the segment:

$$[\gamma, \gamma + 1] \to [D_{\min}, D_{\max}]$$
$$x_1 \mapsto (\omega_{\gamma+1} - \omega_\gamma)x_1 + D_{\max} - \gamma\omega_{\gamma+1} - (1+\gamma)\omega_\gamma$$

We restrict our operator defined by (1.1) to $L^2(D_{t,k})$ with Dirichlet boundary conditions. We denote it by $H_{t,k}$. It is a self-adjoint operator and is considered as the Friedrichs extension associated to the following positive and symmetric quadratic form:

$$\mathcal{H}_{t,k}[u,u] = \int_{D_{t,k}} \rho^{-1} \nabla u \nabla \overline{u}\, dx; \quad u \in C_0^\infty(D_{t,k}).$$

As H is an elliptic operator, $H_{t,k}$ is with compact resolvent hence, it has a purely-discrete spectrum. Let us denote its eigenvalues by

$$0 < E_0(t,k) \le E_1(t,k) \le \cdots \le E_{n-1}(t,k) \le E_n(t,k) \le \cdots.$$

Notice that for any $t \ge 0$ we have $H_0^1(D_{t,k}) \subset H_0^1(D_{0,k})$ hence we have the following relation for the forms in $L^2(D_{0,k})$; $\mathcal{H}_{0,k} \le \mathcal{H}_{t,k}$. This entails that for any $n \in \mathbb{N}$ one gets that $E_n(0,k) \le E_n(t,k)$. The following Lemma gives a lower bound of distance between those eigenvalues.

Lemma 3.2. *For any* $(k \in (2\mathbb{N}+1)$ *and* $n \in \mathbb{N}$*, one has*

$$E_n(t,k) - E_n(0,k) \ge \frac{2t}{\rho_0 \cdot D_{\max}^2}.$$

Proof: For $\lambda << 1$, we set $\widetilde{D}_{\lambda,k} = \{(x_1,x_2);\, x_1 \in \Lambda_k,\ 0 < x_2 < (1-\lambda)f(x_1)\}$. We notice that $D_{0,k} = \widetilde{D}_{0,k}$. Let

$$\psi_\lambda : D_{0,k} \to \widetilde{D}_{\lambda,k}$$
$$(x_1,x_2) \mapsto (x_1,(1-\lambda)x_2).$$

Now consider the following eigenvalue problem on $L^2(\widetilde{D}_{\lambda,k})$.

$$H_{\widetilde{D}_{\lambda,k}} \varphi_{n,\lambda} = \widetilde{E}_n(\lambda,k)\varphi_{n,\lambda}; \quad \varphi_{n,\lambda} \in H_0^1(\widetilde{D}_{\lambda,k}) \tag{3.14}$$

and the quadratic form

$$\widetilde{\mathcal{Q}}_\lambda = \int_{\widetilde{D}_{\lambda,k}} u\big(\psi_\lambda^{-1}(x)\big)\overline{v}\big(\psi_\lambda^{-1}(x)\big)\,dx = \int_{D_0} u(x)\overline{v}(x)(1-\lambda)\,dx$$
$$= (1-\lambda)\langle u, v\rangle, \text{ with domain } \mathrm{H}_0^1(\mathrm{D}_0). \tag{3.15}$$

This leads to a scalar product in $L^2(\widetilde{D}_{\lambda,k})$. Let us consider the following form with domain $H_0^1(D_0)$ which corresponds to H restricted to $\widetilde{D}_{\lambda,k}$;

$$\widetilde{\mathcal{H}}_\lambda = \int_{\widetilde{D}_{\lambda,k}} \rho^{-1}\big(\psi_\lambda^{-1}(x)\big)\nabla u\big(\psi_\lambda^{-1}(x)\big)\nabla\overline{v}\big(\psi_\lambda^{-1}(x)\big)\,dx$$
$$= \int_{D_0}\left(\frac{1}{\varrho}\partial_{x_1}u\partial x_1\overline{v} + \frac{1}{\rho_0(1-\lambda)^2}\partial_{x_2}u\partial_{x_2}\overline{v}\right)(1-\lambda)\,dx.$$

This results in the following operator

$$\widetilde{H}_\lambda = -(1-\lambda)\left(\partial_{x_1}\frac{1}{\varrho}\partial_{x_1} + \frac{1}{\rho_0(1-\lambda)^2}\partial_{x_2}^2\right)$$

with domain $H_0^1(D_0)$. This transform the Eq. (3.14) as follows

$$-(1-\lambda)\left(\partial_{x_1}\frac{1}{\varrho}\partial_{x_1} + \frac{1}{\rho_0(1-\lambda)^2}\partial_{x_2}^2\right)\varphi_{n,\lambda} = \widetilde{E}_n(\lambda,k)(1-\lambda)\varphi_{n,\lambda}; \tag{3.16}$$

which it self yields the following equation

$$-\left(\partial_{x_1}\frac{1}{\varrho}\partial_{x_1} + \frac{1}{\rho_0(1-\lambda)^2}\partial {x_2}^2\right)\varphi_{n,\lambda} = \widetilde{E}_n(\lambda,k)\varphi_{n,\lambda}. \tag{3.17}$$

So, we deal with an analytic family of operators

$$\ddot{H}_\lambda = -\left(\partial_{x_1}\frac{1}{\varrho}\partial_{x_1}\right) - \frac{1}{\rho_0(1-\lambda)^2}\partial_{x_2}^2;\ |\lambda| \ll 1. \tag{3.18}$$

With domain $H_0^1(D_0)$.

When we dervive both sides of the analogue of the Eq. (3.18) for the forms with respect to λ, one entails that for any $n \in \mathbb{N}^*$

$$\widetilde{E}'_n(\lambda,k) = \langle \ddot{H}'_\lambda\varphi_{n,\lambda}, \varphi_{n,\lambda}\rangle \tag{3.19}$$
$$\geq \frac{2}{\rho_0(1-\lambda)^3}\langle\partial_{x_2}\varphi_{n,\lambda}, \partial_{x_2}\varphi_{n,\lambda}\rangle \tag{3.20}$$
$$= \frac{2}{\rho_0(1-\lambda)^3}\|\partial_{x_2}\varphi_{n,\lambda}\|^2, \tag{3.21}$$

Using the Poincaré inequality,[4] we obtain that

$$\widetilde{E}_n'(\lambda, k) \geq \frac{4}{\rho_0 \cdot D_{\max}^2 (1-\lambda)^3}; \quad |\lambda| \ll 1. \tag{3.22}$$

As $D_{t,k} \subset \widetilde{D}_{\frac{t}{D_{\max}}}$, we get

$$E_n(t,k) \geq \widetilde{E}_n\left(\frac{t}{D_{\max}}, k\right). \tag{3.23}$$

Taking into account the fact that $D_{0,k} = \widetilde{D}_{0,k}$ we get that for any $n \in \mathbb{N}$,

$$E_n(0,k) = \widetilde{E}(0,k).$$

This and (3.23) yield that

$$\begin{aligned} E_n(t,k) - E_n(0,k) &\geq \widetilde{E}_n\left(\frac{t}{D_{\max}}, k\right) - \widetilde{E}_n(0,k) \\ &\geq \int_0^{\frac{t}{D_{\max}}} E_n'(\lambda, k) d\lambda. \\ &\geq \int_0^{\frac{t}{D_{\max}}} \frac{4}{\rho_0 \cdot D_{\max}^2 (1-\lambda)^3} d\lambda \\ &= \frac{2}{\rho_0 \cdot (D_{\max} - t)^2} - \frac{2}{\rho_0 \cdot D_{\max}^2} = \frac{2 \cdot t}{\rho_0 \cdot D_{\max}^2}. \end{aligned}$$

□

Theorem 3.3. *(Feynman Hellman Theorem) Let $H(s)$ be a one parameter family of self-adjoint operators for $s \in I$, a neighborhood of zero supposes that $H(s)$ has a simple eigenvalue $E(s) \in C^1(I)$ with eigenfunction $\phi(s) \in C^1(I)$. We have*

$$\frac{dE}{ds}(s) = \left\langle \phi(s), \left(\frac{dH}{ds}(s)\right)\phi(s)\right\rangle.$$

Proof: Using the eigenfunction equation one gets that for any $s \in I$

$$\langle \phi(s), \big(E(s) - H(s)\big)\phi(s)\rangle = 0.$$

Differentiate each side of the last equation. This, with the fact that

$$\left\langle \frac{d\phi}{ds}(s), \big(E(s) - H(s)\big)\phi(s)\right\rangle = 0,$$

and similarly for the conjugate term. As $\|\phi\| = 1$ one gets the result from the term involving $\frac{d}{ds}(H(s) - E(s))$. □

Let

$$D^b_{\lambda,k} = \{(x_1, x_2);\ x_1 \in \Lambda_k,\ 0 < x_2 < D_{\max} - \lambda b(x_1)\}.$$

With, $b\colon \Lambda_k \to [D_{\min}, D_{\max}]$ supported in Λ_k and once differentiable.

Let $H^b_{\Lambda,k}$, be the operator given by (1.1) restricted to $D^b_{\lambda,k}$, with Neumann boundary conditions on the part in $\partial\Lambda_k \times [0, D_{\max}]$ and Dirichlet boundary conditions for the remaining part. Using an analogous map as ψ_λ, one transforms $D^b_{\lambda,k}$ to $D_{0,k} = \Lambda_k \times [0, D_{\max}]$. As done previously, this produces a family of operators on $L^2(D_0)$, having a sequence $(E^b_n(\lambda, k))_{n\in\mathbb{N}}$, of purely-discrete spectra. The following result deals with the first eigenvalue.

Proposition 3.4. *There exists* $K > 0$ *such that*

$$\left(E^b_0\right)'(0, k) \geq K \cdot \frac{1}{|\Lambda_k|} \int_{\Lambda_k} b(x_1)\, dx_1. \tag{3.24}$$

Proof: Let us consider the trivial function

$$\varphi_\lambda(x_1, x_2) = \left(x_1, \frac{D_{\max} - \lambda b(x_1)}{D_{\max}} x_2\right);$$

which transforms $D_{0,k}$ to $D^b_{\lambda,k}$. By an analogous way as we did previously for the proof of Lemma 3.2 we get the following form on $L^2(D_{0,k})$

$$\mathcal{Q}_\lambda[u, v] = \int_{D_{0,k}} u(x)\overline{v}(x)\frac{D_{\max} - \lambda b(x_1)}{D_{\max}} dx,$$

and

$$\begin{aligned}
\mathcal{H}_\lambda[u, v] &= \int_{D^b_{\lambda,k}} \rho^{-1}\big(\varphi_\lambda^{-1}\big)(x)\nabla u\big(\varphi_\lambda^{-1}(x)\big)\nabla\overline{v}\big(\varphi_\lambda^{-1}(x)\big)\, dx;\\
&= \int_{D_{0,k}} \left(\frac{1}{\varrho(x)}\left(\partial_{x_1} u(x)\partial_{x_1}\overline{v}(x) + \frac{\lambda b'(x_1)x_2}{D_{\max} - \lambda b(x_1)}(\partial_{x_1} u\partial_{x_2}\overline{v} + \partial_{x_2} u\partial_{x_1}\overline{v})(x)\right)\right.\\
&\quad \left. + \frac{1}{\rho_0}\left(\frac{D_{\max}}{D_{\max} - \lambda b(x_1)}\right)^2 \partial_{x_2} u\partial_{x_2}\overline{v}(x)\right)\frac{D_{\max} - \lambda b(x_1)}{D_{\max}}\, dx
\end{aligned}$$

acting on $H^1(\Lambda_k) \otimes H^1_0(0, D_{\max})$. The associated operator which we denote by H_λ has a unique ground state, u_λ satisfying

$$H_\lambda u_\lambda = E_0(\lambda, k) M_\lambda u_\lambda, \tag{3.25}$$

here M_λ is the multiplication by $\dfrac{D_{\max} - \lambda b(x_1)}{D_{\max}}$. We set $v_\lambda = M_\lambda^{\frac{1}{2}} u_\lambda$ which transforms (3.25) on

$$M_\lambda^{-\frac{1}{2}} H_\lambda M_\lambda^{-\frac{1}{2}} v_\lambda = E_0^b(\lambda, k) v_\lambda.$$

This gives a new eigenvalue problem for $\breve{H}_\lambda \equiv M_\lambda^{-\frac{1}{2}} H_\lambda M_\lambda^{-\frac{1}{2}}$. $\breve{H}_\lambda$ can be seen as the self-adjoint operator associated with the quadratic form

$$\breve{\mathcal{H}}_\lambda[u, v] = \mathcal{H}_\lambda\big[M_\lambda^{-\frac{1}{2}} u, M_\lambda^{-\frac{1}{2}} u\big].$$

Using Feynman Hellman Theorem, one gets that

$$E_0'(0, k) = (\breve{\mathcal{H}}_0)'[u_0, u_0].$$

Here u_0 is the unique normalized ground state of $\breve{\mathcal{H}}_0$.

Using the min-max principle and the fact

$$E_0^b(\lambda, k) = \inf_{\{u \in H^1(\Lambda_k) \otimes H_0^1(0, D_{\max}),\ \|u\|=1\}} \breve{\mathcal{H}}_\lambda[u, u],$$

one gets that u_0 has to minimize $h[u, u]$ where h is the form associated with the Laplacian on the domain $D_{0,k}$. We notice that here one deals with the bottom of the spectrum. Such problem is studied in Refs. 5, 29 for acoustic operators, and in Ref. 30 for more general divergence form operators. This gives that u_0 is given in term of the ground state for the free Laplacien operator which itself is already known and given by

$$u_0(x_1, x_2) = \sqrt{\frac{2}{D_{\max}|\Lambda_k|}} \sin\left(\frac{\pi x_2}{D_{\max}}\right). \tag{3.26}$$

This yields that

$$\begin{aligned} E_0'(0, k) &= \int_{D_0} \rho_\omega^{-1} \left(\frac{b'(x_1)}{D_{\max}} (\partial_{x_1} u_0(x))^2 + \frac{2b'(x_1)x_2}{D_{\max}} (\partial_{x_1} u_0 \partial_{x_2} u_0)(x)\right) \\ &\quad + \frac{2b(x_1)}{\rho_0 \cdot D_{\max}} (\partial_{x_2} u_0)^2(x)\, dx \\ &\geq \frac{1}{\rho_0 \cdot D_{\max}} \int_{D_0} 2b(x_1)(\partial_{x_2} u_0)^2(x)\, dx \\ &\geq K \frac{1}{|\Lambda_k|} \int_{\Lambda_k} b(x_1)\, dx_1. \end{aligned}$$

□

The following result sets out to estimate the remainder term in the Taylor expansion of $E_0^b(\lambda, k)$. This is related to the Taylor expansion of $\breve{\mathcal{H}}_\lambda$. It is based on the study of an analytic family of perturbation and given on a more general context in Sec. 7 of Ref. 18.

Proposition 3.5. *Refs. 18, 26 Under our assumption there exits $\kappa = \kappa(D_{\max}, b)$ and $K > 0$ such that for any Λ_k such $k \geq \frac{D_{\max}}{\sqrt{3}}$ and $0 \leq \lambda \leq \frac{\kappa}{k^2}$ we have*

$$\left|E_0^b(\lambda, k) - E_0 - \lambda\big(E_0^b\big)'(0, k)\right| \leq \frac{K\pi^2}{4\kappa^2} \cdot k^2 \cdot \lambda^2. \tag{3.27}$$

Here E_0 is the lowest eigenvalue of the operator H_ω and given by (2.10)

The idea of the proof of the last proposition as it was said above is based on the Taylor expansion of $\check{\mathcal{H}}_\lambda$, precisely of the n-th Taylor coefficient $(\check{\mathcal{H}}_0)^{(n)}$ of $\check{\mathcal{H}}_\lambda$ at 0.

4. THE PROOF OF THEOREM 2.1

As is stated, this section is devoted to the proof of Theorem 2.1. Let us start by the lower bound.

4.1. The Lower Bound

For $k \in (2\mathbb{N}+1)$ large enough let us suppose that for any $\gamma \in [-\frac{k}{2} - 1, \frac{k}{2} + 1] \cap \mathbb{Z}$, we have $\omega_\gamma = 0$ then we get

$$E_0\big(H^D_{\Lambda_k}(\omega)\big) = \inf \Sigma\big(H^D_{\Lambda_k}(\omega)\big).$$

We recall that we denote by $H^D_{\Lambda_k}(\omega)$ is the operator (1.1) restricted to $D_{\Lambda_k}(\omega) = \Lambda_k \times ((0, D_{\max}) \cap D(\omega))$ with Dirichlet boundary condition also on the vertical part of the domain.

Let $0 < \varepsilon < d$. We set $D_\varepsilon = (-\frac{k}{2}, \frac{k}{2}) \times (0, D_{\max} - \varepsilon) \subset D_0$. We have

$$\inf \Sigma(H_{D_\varepsilon}) = \frac{\pi^2}{\rho_0(D_{\max} - \varepsilon)^2} + \frac{\pi^2}{c_0 k^2} = E_0(\varepsilon); \tag{4.28}$$

for some constant c_0 such that $\mu_0 \leq c_0 \leq \rho_0$. Let us assume that for any $\gamma \in [-\frac{k}{2} - 1, \frac{k}{2} + 1] \cap \mathbb{Z},\ \omega_\gamma \in (0, \varepsilon)$ then we have

$$D_\varepsilon \subset D_{\Lambda_k}(\omega) \subset D_\omega.$$

So,

$$H_\omega \leq H^D_{\Lambda_k}(\omega) \leq H_{D_\varepsilon},$$

and consequently we get

$$E_0(H_\omega) \leq E_0\big(H^D_{\Lambda_k}(\omega)\big) \leq E_0(\varepsilon). \tag{4.29}$$

If we take $\varepsilon = \frac{1}{k^2}$, the Eq. (4.28) yields

$$E_0(\varepsilon) \leq E_0 + \frac{c}{k^2}.$$

Here $c = \frac{\pi^2}{\rho_0}(1 + \frac{2}{D(D-1)^2})$.

Using Eq. (2.11) one gets,

$$\begin{aligned} N(E_0+\varepsilon) &\geq \frac{1}{(2k+1)} \cdot \mathbb{P}\big\{E_0(H_{\Lambda_k}(\omega)\big) \leq E_0 + \varepsilon\big\} \\ &\geq \frac{1}{(2k+1)} \cdot \mathbb{P}\big\{E_0\big(H^D_{\Lambda_k}(\omega)\big) \leq E_0(\varepsilon)\big\} \\ &= \frac{1}{(2k+1)} \cdot \mathbb{P}\{\Lambda_k(\omega) \subset D_\varepsilon\} \\ &= \frac{1}{(2k+1)} \cdot \mathbb{P}\left\{\forall \gamma \in \left[-\frac{k}{2} - 1, \frac{k}{2} + 1\right] \cap \mathbb{Z};\ \omega_\gamma \leq \varepsilon\right\} \quad (4.30) \\ &= \frac{1}{(2k+1)} \cdot \mathbb{P}\{\omega_0 \in (0, \varepsilon)\}^{(k+2)}. \quad (4.31) \end{aligned}$$

The proof is ended by taking into account assumption **(A.1)** and computing the limit for $\varepsilon = \dfrac{1}{k^2}$.

4.2. The Upper Bound

The proof of the upper bound is based on the use of tools stated on the previous section and on a probabilistic technique known as the large deviation argument.

Let $H^N_{\Lambda_k}(\omega)$ be the operator defined by (1.1) restricted to $D_\omega \cap (\Lambda_k \times (0, D_{\max}))$ with Neumann boundary conditions on the vertical parts of $D_\omega \cap (\partial\Lambda_k \times (0, D_{\max}))$ and Dirichlet boundary conditions for the remaining part. For the choosing boundary conditions one has

$$H^N_{\Lambda_k}(\omega) \leq H_{\Lambda_k}(\omega). \quad (4.32)$$

Lemma 4.1. *There exits $c > 0$, $K_2 > 0$ such that for $a > 0$*

$$\mathbb{P}\left\{E_0\big(H^N_{\Lambda_k}(\omega)\big) \leq E_0 + \frac{a}{k^2}\right\} \leq c \cdot e^{-k\frac{(m-K_2\sqrt{a})^2}{c}}$$

Proof: For $\psi \in C^1_0(-\frac{1}{2}, \frac{1}{2})$ such that $\psi(0) = 1$ and for any $x \in (-\frac{1}{2}, \frac{1}{2})$ one has

$$0 \leq \psi(x) \leq 1 - |x|.$$

We set

$$b_\omega(x_1) = \sum_{\gamma\in\Lambda_k} \omega_\gamma \psi(x_1 - \gamma).$$

Taking the same notation as in Sec. 3, we get that $D^{b_\omega}_{\lambda,k}$ having the same rate as $D_{0,k}$ with smooth corners and with the property that $D^{b_\omega}_{\lambda,k} \subset D_{\lambda,k}$. Indeed using the properties of ω_γ, we get that for $x_1 \in (0, \frac{1}{2})$ we have

$$-\omega_0(1 - x_1) - \omega_1 x_1 \le -\omega_0(1 - |x_1|)$$

and for $x_1 \in (-\frac{1}{2}, 0)$ we have

$$-\omega_0(1 + x_1) + \omega_{-1} x_1 \le -\omega_0(1 - |x_1|).$$

So using the notation of Remark 3.1 one gets

$$f_\omega \le D_{\max} - b_\omega.$$

Thus if we note by $E^{b_\omega}_0(\lambda, k)$ the first eigenvalue of the operator (1.1) restricted to $D^{b_\omega}_{\lambda,k}$ and take into account (4.32) one gets that

$$E_0(H^N_{\Lambda_k}(\omega)) \ge E^{b_\omega}_0(\lambda, k);\ \ \forall\lambda \in (0, 1). \tag{4.33}$$

Using (3.24), one gets that there exists $K_1 > 0$ such that

$$\frac{dE^{b_\omega}_0}{d\lambda}(\lambda, k)(0) \ge \frac{2\pi^3}{D^3_{\max}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi(x)\,dx \cdot \left(\frac{1}{|\Lambda_k|}\sum_{\gamma\in\Lambda_k} \omega_\gamma\right). \tag{4.34}$$

$$= K_1 \cdot \left(\frac{1}{|\Lambda_k|}\sum_{\gamma\in\Lambda_k} \omega_\gamma\right). \tag{4.35}$$

We recall that from Proposition 3.5, we have

$$\left|E^{b_\omega}_0(\lambda, k) - E_0 - \lambda\big(E^{b_\omega}_0\big)'(0, k)\right| \le \frac{K\pi^2}{4\kappa^2} \cdot k^2 \cdot \lambda^2. \tag{4.36}$$

So, if we assume that for $a \le \dfrac{\pi^2 K}{4}$ we have

$$E^{b_\omega}_0(\lambda, k) \le E_0 + \frac{a}{k^2}, \tag{4.37}$$

then Eqs. (4.35), (4.36) and (4.37) implies that

$$\lambda \cdot \big(E^{b_\omega}_0\big)'(0, k) \le \frac{K\pi^2}{4\kappa^2} \cdot k^2 \cdot \lambda^2 + \frac{a}{k^2}.$$

Tacking, $\lambda = \frac{t\kappa}{k^2}$, one gets

$$\left(E_0^{b_\omega}\right)'(0,k) \leq \frac{K\pi^2 \cdot t}{4\kappa} + \frac{a}{\kappa t}; \tag{4.38}$$

for any $0 \leq t \leq 1$. Optimizing (4.38), with respect to t one gets that $t_0 = \frac{2\sqrt{a}}{\pi\sqrt{K}} < 1$. Taking into account (4.35) we get that for $K_3 = \frac{\sqrt{K}\pi}{2\kappa}$

$$\begin{aligned}
\mathbb{P}\left\{E_0^{b_\omega}(\lambda,k) \leq E_0 + \frac{a}{k^2}\right\} &\leq \mathbb{P}\{(E_0^{b_\omega})'(0,k) \leq K_3\sqrt{a}\} \\
&\leq \mathbb{P}\left\{\frac{1}{(2k+1)}\sum_{\gamma\in\Lambda_k}\omega_\gamma \leq K_2K_1\sqrt{a}\right\} \\
&\leq \mathbb{P}\left\{\left|\frac{1}{(2k+1)}\sum_{\gamma\in\Lambda_k}\omega_\gamma - m\right| \geq m - K_2K_1\sqrt{a}\right\} \\
&\leq c\cdot e^{-k\cdot\frac{(m-K_2\sqrt{a})^2}{c}}.
\end{aligned} \tag{4.39}$$

The last estimation is due to a large deviation argument[(6)] where we take a small such that $0 \leq \sqrt{a} \leq \frac{m}{K_1\cdot K_2}$.

The proof of Lemma 4.1 is now ended by taking into account (4.33) and (4.39). □

Let us recall the following properties from that from (2.11) one deduces

$$N(E_0+\varepsilon) \leq \frac{1}{2k+1}\mathbb{E}\left(N\left(H_{\Lambda_k}^N(\omega), E_0+\varepsilon\right)\right).$$

Using the Weyl estimate one gets that a lower bound on $\frac{1}{2k+1}\cdot N(H_{\Lambda_k^N(\omega)}, E_0+\varepsilon)$ by $K = c_d(E_0+\varepsilon)^{\frac{1}{2}}$. So,

$$N(E_0+\varepsilon) \leq K\int_{\{\omega, E_0(H_{\Lambda_k}^N(\omega))\leq E_0+\varepsilon\}} d\mathbb{P} = K\cdot\mathbb{P}\left\{E_0\left(H_{\Lambda_k}^N(\omega)\right) \leq E_0+\varepsilon\right\}.$$

For $\varepsilon = \frac{a}{k^2}$ in Lemma 4.1, one gets

$$N(E_0+\varepsilon) = N\left(E_0 + \frac{a}{k^2}\right) \leq c\cdot e^{-\frac{(m-K_2\sqrt{a})^2}{\sqrt{\varepsilon}c}}.$$

The proof of the upper bound is then ended by taking the double logarithm of the last equation.

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REFERENCES

1. M. Aizenman and Molchanov, Localization at large disorder and at extreme energies:an elementary derivation. *Commun. Math. Phys.* **157**:245–278 (1993).
2. R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators.* (Birkhäuser Boston, Basel, Berlin, 1990).
3. D. Damanik and P. Stollmann, Multiscale analysis implies strong dynamical localization. *Geom. Funct. Anal.* **26**(3):411–428 (2001).
4. R. Dautray and J-L. Lions, *Mathematical Analysis and Numerical Methods for Sience and Technology.* vol 2 (Springer-Verlag, Berlin, 1988).
5. E. B. Davies and L. Parnovski, Trapped modes in acoustic waveguides. *Qua. Jour. Mech. Appl Math.* **51**(3):477–492 (1998).
6. A. Dembo and O. Zeitouni. *Large Deviation and Applications* (Jones and Bartlett Publishers, Boston, 1992).
7. J. Fröhlich and T. Spencer, Absence of diffusion in the Anderson tight binding model. *Commu. Math. Phys.* **88**:151–184 (1983).
8. H. von Dreifus and A. Klein, A new proof of localization in the Anderson tight binding model. *Commun. Math. Phys.* **124**:285–299 (1989).
9. P. Duclos and P. Exner, Curvature-induced Bound States in Quantum waveguides in two and three dimensions. *Rev. Math. Phy.* **37**:4867–4887 (1989).
10. D. Borisov and P. Exner, Exponential splitting of bound states in a waveguide with a pair of distant windows. *J. Phys. A* **37**(10):3411–3428 (2004).
11. P. Exner, P. Šeba, M. Tater, and D. Vaněk, Bound states and scattering in quantum waveguides coupled laterally through a boundary window. *J. Math. Phys.* **37**(10):4867–4887 (1996).
12. P. Exner and S. A Vugalter, Asymptotic Estimates for bound states in quantum waveguide coupled laterally through a boundary window. *Ann. Inst. H. Poincaré Phy. Théor.* **65**:109–123 (1996).
13. A. Figotin and A. Klein, Localization of classical waves i: acoustic waves. *Commun. Math. Phys.* **180**:439–482 (1996).
14. J. Fröhlich, F. Martinelli, E. Scoppola and T. Spencer, A Constructive proof of localization in the Anderson tight binding model. *Commu. Math. Phys.* **101**:21–46 (1985).
15. F. Germinet and S. De Bièvre, Dynamical localization for discrete and continous random Schrödinger operators. *Commu. Math. Phys.* **194**(2):323–341 (1998).
16. F. Germinet and A. Klein, Bootstrap multiscale analysis and localization in random media. *Commu. Math. Phys.* **222**(2):415–448 (2001).
17. S. John, The localization of light in photonic gaps and localization. *NATO ASI Serie B:Physical* 308–321 (1993).
18. T. Kato, *Perturbation Theory for linear operators* (Springer-Verlag, New York, 1980).
19. A. Klein, J. Lacroix, and A. Speis, Athanasios, Localization for the Anderson model on a strip with singular potentials. *J. Funct. Anal.* **94**(1):135–155 (1990).

20. A. Klein, J. Lacroix and A. Speis, Regularity of the density of states in the Anderson model on a strip for potentials with singular continuous distributions. *J. Statist. Phys.* **57**(1–2):65–88 (1989).
21. A. Klein and A. Speis, Klein, Abel; Speis, Athanasios Smoothness of the density of states in the Anderson model on a one-dimensional strip. *Ann. Physics* **183**(2):352–398 (1988).
22. W. Kirsch, Random Schrödinger operators. A course. In: *Schrödinger operators, Sønderborg 1988, Lecture Notes in Phys.* ***345*** (Springer, Berlin, 1989) pp. 264–370.
23. W. Kirsch, Wegner estimates and Anderson localization for alloy-type potentials. *Math. Z.* **221**:507-512 (1996).
24. W. Kirsch, P. Stollmann and G. Stolz: Anderson localization for random Schrodinger operators with long range interactions. *Commu. Math. Phys.* **195**:495–507 (1998).
25. W. Kirsch and F. Martinelli, On the Spectrum of Schrödinger operators with a random potential. *Commu. Math. Phys.* **85**:329–350 (1982).
26. F. Kleespies and P. Stollmann, Lifshitz asymptotics and localization for random quantum waveguides. *Rev. Math. Phy.* **12**:1345–1365 (2000).
27. F. Klopp, Localization for some continus random Schrödinger operators. *Commun. Math. Phys.* **167**:553–569 (1995).
28. F. Klopp, Internal Lifshits tails for random perturbations of periodic Schrödinger operators. *Duke Mathematical J.* **98**(2):335–396 (1999).
29. S. Kondej and I.Veselić, Spectral gap of segments of periodic waveguides. *Let. Math. Phy.* (2006).
30. S. Kondej and I.Veselić, The low-lying spectrum of periodic quantum waveguides. in prepation.
31. S.M. Kozlov, Conductivity of two-dimrnsional random media. *Russian Math. Surveys* **34**(4):168-169 (1979).
32. P. Kuchment, *Floquet Theory for Partial Differential Equations* volume 60 of Operator Theory: Advances and applications (Birkhäser, Basel, 1983).
33. J. T. Londergan, J. P. Carini and D. P. Murdock, *Binding and Scattering in Two-dimensional Systems* (Springer, Berlin, 1999).
34. I. Lifshitz, Structure of the energy spectrum of impurity bands in disordered solid solutions. *Soviet Phy. JETP* **17**:1159–1170 (1963).
35. H. Najar, Lifshitz tails for random acoustic operators. *J. Math. Phys.* **44**(4):1842–1867 (2003).
36. H. Najar, Asymptotic behavior of the integrated density of states of acoustic operator with long range random perturbations. *J. Stat. Phys.* **115**(4):977–996 (2003).
37. H. Najar, Non-Lifshitz tails at the spectrum bottom of some random operator. submitted
38. H. Najar, 2-Dimensional localization of acoustic waves in random perturbation of periodic media. *J. Math. Ana. App.* **322**(I):1–17 (2006).
39. H. Najar, Non-Lifshitz tails at the spectrum bottom of some random operators. submitted
40. L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators* (Springer-Verlag, Heidelberg, 1992).
41. M. Reed and B. Simon, *Methods of Modern Mathematical Physics* Vol. IV: Analysis of Operators (Academic, Press, 1978).
42. P. Stollmann, *Caught by Disorder Bounded States in random Media* (Birkhäuser, 2001).
43. I. Veselić, *Localization for random perturbation of periodic Schrödinger operators with regular Floquet eigenvalues*. Technical report Unive. Bochum.